Robust Model Predictive Control of Systems by Modeling Mismatched Uncertainty

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Abstract: This study addresses to the robustness of model predictive control in the presence of the mismatched uncertainty, e.g. disturbance, noise and parameter variations. Model predictive control is solved online and its control action is fed to the real system with the additional control action that is required to maintain the controlled trajectories in a simple uncertainty tube in practice where the center of the aforementioned tube is the trajectory of the nominal model. For this purpose, a sliding mode controller as variable control structure is designed taking the difference between the real system and nominal system into consideration. The stability of the overall system is proven taking the modeling error on the uncertainty model into account.

Keywords: Mismatched uncertainty, model predictive control, robustness, sliding mode control, tubes, uncertain systems.

1. INTRODUCTION

Model predictive control (MPC), an extension of optimal control, is a model-based control approach which is able to deal with the constraints on the states and inputs (Mayne et al., 2000; Kayacan et al., 2014). The concept of MPC is to compute state trajectories in a finite horizon by minimizing a cost function consisting of all states and inputs online (Kayacan et al., 2015c). After solving optimization problem, only the first element of the generated input sequence is fed to the system (Kayacan et al., 2015a). Then, the finite horizon is shifted over time for the next sampling time (Morari and Lee, 1999; Qin and Badgwell, 2003). The same procedure is repeated again when new measurements or estimates are obtained.

The robust stability of MPC is obtained only if the nominal model is inherently robust without estimation errors, uncertainties and parameter variations. However, systems are always subjected to disturbances in real life (Grimm et al., 2004). Inasmuch as unmodeled uncertainties may result in instability of systems, robust MPC method has become a crucial topic and been developed due to the emphasis of the handling uncertainties in practice (Calafiore and Fagiano, 2013; Yan and Wang, 2014). One of the most significant method is the tube-based approach proposed for state and output feedback MPC (Langson et al., 2004; Mayne et al., 2005, 2006). In these previous studies, the uncertainty has not been modeled and the additional control action is designed in which the uncertainty error is multiplied by a state feedback controller and then the product is fed to the real system. Furthermore, an integral sliding manifold (ISM) has been proposed instead of a state feedback controller (Rubagotti et al., 2011).

This paper focuses on robust model predictive control for systems with mismatched uncertainties. The main contribution of this study is to formulate a tube-based approach by modeling the uncertainty structure. In order to handle the uncertainties in the real system, sliding mode control method, which is inherently robust to uncertainties, is used to design a controller by taking the difference between the nominal model and real system. The stability of the overall system is proven by using a Lyapunov function.

This paper is organized as follows: The previous works are summarized in Section 2. The controller is designed and the control structure is presented in Section 3. The system is presented in Section 4. The simulation results are given in Section 5. Finally, a brief conclusion is given in Section 6.

2. PROBLEM FORMULATION

The discrete-time linear time-invariant system model is represented by

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector and $w \in \mathbb{R}^n$ is the disturbance vector. The constraints on the state and input are denoted by

$$x \in \mathbb{X}, \quad u \in \mathbb{U},$$
 (2)

where $\mathbb{X} \subset \mathbb{R}^n$ is closed, $\mathbb{U} \subset \mathbb{R}^m$ is compact and they have their own origin in their own interior. It is assumed that the disturbance w is bounded

$$w \in W$$
 (3)

where W is compact and includes the origin.

The nominal system respecting the system (1) is denoted by

$$\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k \tag{4}$$

It is assumed that the controller \bar{u}_k is equal to $K\bar{x}_k$ where $K \in \mathbb{R}^{m \times n}$ denotes the coefficients of the controller and the closed-loop system $A^K = A + BK$ is stable. The disturbance set denoted

^{*} The information, data, or work presented herein was funded in part by the Advanced Research Projects Agency-Energy (ARPA-E), U.S. Department of Energy, under Award Number DE-AR0000598

by Z for the real system model $x_{k+1} = A^K x_k + w$ fulfills the following condition,

$$A^K X \oplus W \subseteq Z \tag{5}$$

where \oplus is Minkowski set addition, the disturbance set Z is invariant and the origin of the real system.

In order to control the real system with mismatched uncertainty in (1), a tube-based MPC approach was proposed in (Mayne and Langson, 2001) as below.

Proposition 1. It is presumed that $x \in \bar{x} \oplus Z$ where Z is the invariant disturbance for $x_{k+1} = Ax_k + Bu_k + w$ and $\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k$. If the control input to the real system $u(\bar{u}, x, \bar{x}) = \bar{u}(\bar{x}) + K(x - \bar{x})$, then $x_{k+1} \in \bar{x}_{k+1} \oplus Z \ \forall w \in W$.

This proposition clearly expresses that the feedback controller $u(\bar{u}, x, \bar{x}) = \bar{u}(\bar{x}) + K(x - \bar{x})$ enforces the states x of the real system $x_{k+1} = A^K x_k + w$ to track the states \bar{x} of the nominal system $\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k$. This proposed control method has been called the tube-based approach. In this former tube-based MPC approach, the states of the nominal model are directly fed to MPC and thus MPC does not have any information exchange with the real system (Langson et al., 2004). This structure was criticized due to the fact that it does not take the measurements coming from the real system into consideration. In order to interact MPC with the real-system, a new method that not only MPC but also the nominal model is initialized by the measurements of the real system at each sampling time instant has been proposed for state feedback case (Mayne et al., 2005) as in the following proposition.

Proposition 2. It is presumed that $x \in \bar{x} \oplus Z$ where Z is the invariant disturbance for $x_{k+1} = Ax_k + Bu_k + w$ and $\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k$. If the control input to the real system $u(\bar{u}, x, \bar{x}) = \bar{u}(x) + K(x - \bar{x}(x))$, then $x_{k+1} \in \bar{x}_{k+1} \oplus Z \ \forall w \in W$.

As an alternative method, only initialization of MPC by the measurements of the real system has been proposed in (Kayacan et al., 2015b, 2016). In these studies, all states of the real system are assumed to be measurable. However, this is not feasible in practice since number of sensors are generally less than number of measured variables. For this reason, output feedback MPC has been developed while a Luenberger observer is employed to estimate immeasurable states (Mayne et al., 2006).

In addition to a state feedback controller, an integral sliding manifold (ISM) has been proposed to handle uncertainties (Rubagotti et al., 2011) as in Proposition 3. In this approach, measurements coming from the real system are fed to MPC while the nominal model is not initialized by measurements coming from the real system.

Proposition 3. It is presumed that $x \in \bar{x} \oplus Z$ where Z is the invariant disturbance for $x_{k+1} = Ax_k + Bu_k + w$ and $\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k$. If the control input to the real system $u(\bar{u}, x, \bar{x}) = \bar{u}(x) - ksgn(x - \bar{x})$, then $x_{k+1} \in \bar{x}_{k+1} \oplus Z \ \forall w \in W$.

In these approaches, the stability analysis is proven over the real system due to the unmodeled uncertainty. In this paper, the nonlinear modeling of the uncertainty model is formulated and then the stability analysis is proven over this uncertainty model as distinct from the previous ones.

3. ROBUST MODEL PREDICTIVE CONTROL

In the controller design process, it is assumed that the nominal control input generated by MPC $\bar{u}=Kx$ is able to stabilize the nominal system so that A+BK is stable. Since the real system and the nominal system are not identical, a controller is required to stabilize the uncertainty model defined as the difference between the real system and nominal model. The uncertainty state z is formulated as

$$z = x - \bar{x}(x) \tag{6}$$

where x and \bar{x} are the states of the real system and nominal system.

Proposition 4. It is presumed that $x \in \bar{x} \oplus Z$ where Z is the invariant disturbance for $x_{k+1} = Ax_k + Bu_k + w$ and $\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k$. The uncertainty model is described as a second-order nonlinear model $\ddot{z} = h(z) + v$ where $z = x - \bar{x}(x)$ is the output of the uncertainty model while v is the input of the uncertainty model. If the control input to the real system $u = \bar{u}(x) + v(x, \bar{x}(x))$, then $x_{k+1} \in \bar{x}_{k+1} \oplus Z \ \forall w \in W$.

The feedback control action $v(x, \bar{x}(x))$ in aforementioned proposition, which requires low sensitivity to plant parameter uncertainty and finite-time convergence, will be formulated based on sliding mode control (SMC) theory. The proposed control structure is illustrated in Fig. 1.

The uncertainty model is represented by the second-order non-linear model as follows:

$$\ddot{z} = h(z) + v \tag{7}$$

where v is the control input, z is the output of the system and h(z) is the nonlinear or time-varying dynamics of the system. The system output z is measurable while the system dynamics h(z) is not known. It is assumed that the function h is upper bound by H as follows:

$$\mid h \mid \leq H \tag{8}$$

The tracking error is written as

$$\tilde{z} = z - z_d \tag{9}$$

where \tilde{z} is the tracking error and z_d is the desired trajectory of the system.

A sliding surface is defined to track the desired trajectory of the system as follows:

$$s = \dot{\tilde{z}} + \lambda \tilde{z} \tag{10}$$

where λ is a positive constant and denotes the slope of the sliding surface. As can be seen in (10), the sliding surface

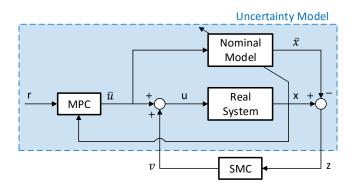


Fig. 1. Control Scheme

basically consists of the uncertainty error and uncertainty error rate

The sliding surface s is linear so that the time derivative of the sliding surface \dot{s} must be zero, i.e. $\dot{s} = 0$. By taking the derivative of (10) by time t, the rate of the sliding surface is obtained as follows:

$$\dot{s} = \ddot{z} + \lambda \dot{z}
= \ddot{z} - \ddot{z}_d + \lambda \dot{z}
= h(z) + v - \ddot{z}_d + \lambda \dot{z}$$
(11)

The equivalent control law v_{eq} that achieves $\dot{s} = 0$ is written as follows:

$$v_{eq} = -h(z) + \ddot{z}_d - \lambda \dot{\bar{z}} \tag{12}$$

Inasmuch as the function h is unknown, the controller law cannot contain the function h so that it must contain an additional term to ensure the sliding motion $\dot{s} = 0$ so that the control input is written as follows:

$$v_{eq} = \ddot{z}_d - \lambda \dot{\tilde{z}}$$

$$v = v_{eq} - k \operatorname{sgn}(s)$$
(13)

where k is a positive coefficient, i.e. k > 0, and adaptive with the following rule below:

$$\dot{k} = \gamma_k \mid s \mid \tag{14}$$

where γ_k is a positive coefficient, i.e. $\gamma_k > 0$.

Theorem 5. The control algorithm, i.e. $u = \bar{u}(x) + v(x,\bar{x}(x))$, consists of the sliding mode controller based on the uncertainty model working in parallel with an MPC controller as shown in Fig. 1. In addition, if the function h is upper bounded by H as seen in (8) and the final value of the controller coefficient k^* in (13) is large enough, i.e. $k^* > H$, then the uncertainty vector z converges asymptotically to zero in finite time and sliding motion will be achieved.

Proof. In order to analyze the system stability, the Lyapunov function is written as follows:

$$V = \frac{1}{2}s^2 + \frac{1}{2\gamma_k}(k - k^*)^2 \tag{15}$$

The time derivative of the Lyapunov function in (15) is written as follows:

$$\dot{V} = s\dot{s} + \frac{\dot{k}}{\gamma_k}(k - k^*) \tag{16}$$

If the adaptation rule for the controller gain k in (14) and the time derivative of the sliding surface in (11) are inserted into (16), it is obtained as follows:

$$\dot{V} = s \left(h - k \operatorname{sgn}(s) \right) + |s| \left(k - k^* \right)$$
(17)

As it is assumed that the function h is upper bounded by H, it is obtained as follows:

$$\dot{V} < |s|(H-k) + |s|(k-k^*)$$

 $< |s|(H-k^*)$ (18)

If the final value of the controller gain k^* is large enough as stated in Theorem 5, i.e. $k^* > H$, then the overall system is stable, i.e. $\dot{V} < 0$.

Remark 1: Since the adaptation rule in (14) is enforced, the final value of the controller gain is unknown that are determined

during the adaptation of the controller gain, and it is able to reach large values to make systems stable. This is a superiority of the proposed approach in this study as distinct from previous studies in which the fine tuning is required.

Remark 2: It is to be noted that the aim in the control structure is to converge the uncertainty to zero. For this reason, the desired values for the uncertainty position, velocity and acceleration are equal to zero, i.e. $z_d = \dot{z}_d = \ddot{z}_d = 0$. Taking the desired values into consideration, the sliding surface in (10) and applied control input (13) become as follows:

$$s = \dot{z} + \lambda z \tag{19}$$

$$v = -\lambda \dot{z} - k \operatorname{sgn}(s) \tag{20}$$

Remark 3: As can be seen in Theorem 5, the sliding mode control method as a variable control structure is inherently robust to uncertainty which is occurred by the modeling error, parameter variations, and noise. For this reason, the designed SMC ensures in low sensitivity to plant uncertainty and finite-time convergence.

4. INVERTED PENDULUM

The system consists of an inverted pendulum mounted on a cart as illustrated in Fig. 2. The cart is restricted to linear motion horizontally where the pendulum is constrained to move in the vertical plane. The control input is the force F that moves the cart horizontally and the outputs are the angular position of the pendulum θ and the horizontal position of the cart on the axis ξ . The variables of the system are represented in Table 1.

Table 1. NOMENCLATURE

θ	Angular position of the pendulum
ξ	Horizontal position of the cart
M	Mass of the cart
m	Mass of the pendulum
I	Inertia moment of pendulum
1	Length to pendulum center of mass
g	Gravitational acceleration
ζ	Viscous friction coefficient for the cart
F	Force applied to the cart

The equations of motion can be written as follows:

Disturbance

$$(M+m)\ddot{\xi} - ml\cos\theta\ddot{\theta} + ml\dot{\theta}^2\sin\theta - \zeta\dot{\xi} = F + d$$
$$-ml\cos\theta\ddot{\xi} + (I+ml^2)\ddot{\theta} - mgl\sin\theta = 0 \tag{21}$$

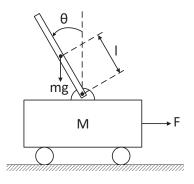


Fig. 2. Inverted Pendulum on a Cart

where θ , ξ , M, m, I, I, g, ζ and d represent respectively the angular position of the inverted pendulum, the horizontal position of the cart, the mass of the cart, the mass of the pendulum, the inertia moment of the pendulum, the length to pendulum center of mass, the gravitational acceleration, the viscous friction coefficient and the disturbance.

The aim of the inverted pendulum on a cart is to keep the pendulum perpendicular to the cart. For this reason, the system is linearized around zero degree angle, i.e. $\theta = 0$, and then the linearized system model is formulated as follows:

$$\dot{x} = Ax + Bu + w$$

$$y = Cx \tag{22}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(I+ml^2)\zeta}{p} & \frac{m^2gl^2}{p} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{ml\zeta}{p} & \frac{mgl(M+m)}{p} & 0 \end{bmatrix}, w = \begin{bmatrix} 0 \\ \frac{I+ml^2}{p} \\ 0 \\ \frac{ml}{p} \end{bmatrix} d,$$

$$B = \begin{bmatrix} 0 \\ \frac{I+ml^2}{p} \\ 0 \\ \frac{ml}{0} \end{bmatrix}, C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}^T, x = \begin{bmatrix} \xi \\ \dot{\xi} \\ \dot{\theta} \\ \dot{\theta} \end{bmatrix}, u = F$$

with $p = (M + m)(I + ml^2) - (ml \cos \theta)^2$.

5. SIMULATION RESULTS

In simulation study, the nonlinear model in (21) is considered as the real system while the linear model in (22) is considered as the nominal model in Fig. 1. The numerical values in this study are $M=0.5~kg,~m=0.2~kg,~l=0.3~m,~I=0.006~m,~g=9.81~m/s^2$. The damping coefficient ζ is set to 0.1 and 0.9 for the nominal system and real system, respectively. The sampling period of the simulation is set to 0.01 second. The following MPC formulation employed at each sampling time is written as follows:

$$\min_{\substack{x(.),u(.)\\\text{subject to}}} \int_{t_k}^{t_k+t_h} \left(\|x(t)\|_Q^2 + \|u(t)\|_R^2 \right) dt + \|x(t_k+t_h)\|_P^2$$

$$\sup_{\substack{x(t)=Ax(t)+Bu(t)\\-\frac{\pi}{2} \le \theta(t) \le \frac{\pi}{2}\\-30 \le u(t) \le 30}} \forall t \in [t_k, t_k+t_p]$$
(23)

where t_h denotes the prediction horizon and is set to 0.4 second, t_k denotes the current time, x denotes the system state, and u denotes the input. The weighting matrices are set to Q = 1, R = 0.01 and P = 1.2.

The values λ and γ_k for the proposed SMC controller in (20)and (14) are respectively set to 10 and 5. Also, the controller gain k for the ISM in (Rubagotti et al., 2011) is set to 5 while the state feedback for the tube-based approach is set to $K = [0,0,-5,-10]^T$. Moreover, due to the fact that the control law contains the sgn function, SMC control method endures high-frequency oscillations,, i.e. chattering. Several methods have been proposed to solve this problem. In this study, the sgn

function is replaced by the following equation to remove the chattering effect:

$$\operatorname{sgn}(s) := \frac{s}{|s| + \delta} \tag{24}$$

where $\delta = 0.1$.

The initial conditions on the states of the system are considered as $[\theta,\dot{\theta}]=[1,0]^T$. A step external force d=1 and a sinusoidal external force $d=\sin 2t$ as disturbance signals are imposed respectively on the system at t=10 second and t=20 second to evaluate the proposed control structure in presence of the mismatched disturbance. The angular position responses are shown in Fig. 3. As seen, the MPC with SMC results in less overshoot and settling time than the other methods. Unlike only MPC applied to the real system, the MPC with SMC is able to stabilize the system while oscillatory behavior and steady-state error are not observed.

The control inputs for the MPC with SMC are shown in Fig. 4. The output of the MPC controller becomes zero while the one of SMC controller does not converge to zero. The reason is that the uncertainty is occurred only due to the linearization,

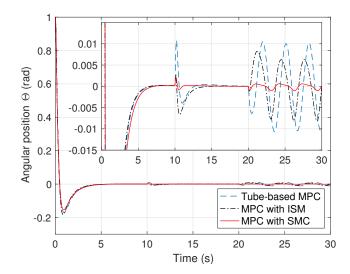


Fig. 3. The angular position responses of the real-system

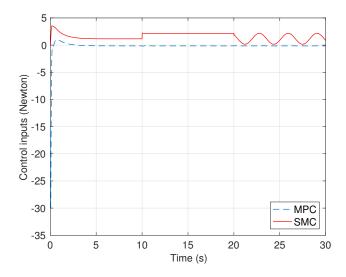
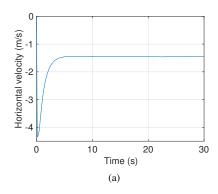
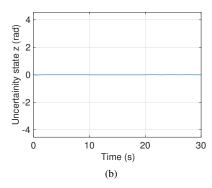


Fig. 4. Control input signals





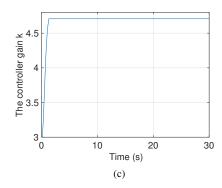


Fig. 5. (a) The horizontal velocity response of the real system (b) The uncertainty state z for the MPC with SMC controller (c) The controller gain

different viscous friction values for the nominal model and real system, and external disturbances. Therefore, the SMC controller generates a control signal to handle the uncertainty problem.

The horizontal velocity is shown in Fig. 5(a). Since the viscous friction values for the nominal model and real system are different, the output of the SMC controller between t=0-10 seconds is around to the force occurred by the difference in viscous friction coefficients. The maximum value of the uncertainty vector w can be equal to 4.54 Newton. As seen in Fig. 5(b), the uncertainty state is around zero. thanks to the proposed MPC with SMC controller, the mismatched uncertainty problem can be disregarded in the proposed control structure.

The adaptation of the controller gain k is resented in Fig. 5(c). The initial condition on the controller is considered as k(0) = 3. Thanks to the adaptation rule, the controller gain is adjusted throughout the simulations. Therefore, it is able to reach a large value which can make the system stable as stated in Remark 1.

6. CONCLUSIONS

The proposed control structure provides robustness to uncertainty with a designed SMC controller. The MPC controller is solved online while the states of the real system are fed and the nominal model is initialized by the the states of the real system at each time instant. The proposed control method, tube-based MPC, is able to provide that the robust asymptotic stability is established in the presence of mismatched uncertainties.. It is practically as simple as the implementation of the conventional MPC.

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